Dynamics of Continuous, Discrete and Impulsive Systems
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THE THEORY OF INFINITESIMAL DIFFEOMORPHISMS: AN INTRODUCTION

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Abstract. Biological and social sciences are hard pressed to apply the laws of Newton due to the multiplicity of parameters affecting the dynamics of such systems; hence the reliance on statistical methods. However, statistical methods do not predict cause-effect relationships. Rather, statistics provides correlations between dynamics of populations. There is a need for a "calculus" that can be used to predict causal relationships in biological and social systems that is based on the operative factors of these systems: complexity.

Complexity has been extensively analyzed by mathematicians and definitively explained by Smale [1,2] through the use of the horseshoe paradigm. The essence of this paradigm is that complexity arises, in its simplest form, from the operation of two dynamics: Stretching and folding. By devising a "calculus" based on stretching and folding we may be better positioned to predict causal relationships in the biological and social sciences. The value of a calculus of stretching and folding to biological and social systems to derive dynamical systems has been presented in [2,3,4,5,6,7]. The calculus of stretching and folding is mathematically formulated in the concept of an infinitesimal diffeomorphism (ID), [4]. In this paper I present a survey of the presently known properties of IDs.

Keywords. Chaos, Algebra, complexity, dynamical systems, dynamical synthesis.

AMS (MOS) subject classification: 37D45.

1 Introduction: How does the study of IDs assist and advance the work of other scientific enterprises?

Infinitesimal diffeomorphisms (IDs) [4] are transformations on a manifold that can closely approximate the solution of a differential equation. However, they are a legitimate subject of analysis in their own right due to (1) their potential application in the biological and social sciences as seen in [7]; (2) their use in the numerical approximation of the solutions of ODEs; (3) their use as closed form diffeomorphisms having complex dynamics that are equivalent to such systems as that of Chua, Lorenz and Rössler thus facilitating the direct study of such systems without the need of ODEs; (4) their independence from the laws of physics; (5) their use in modeling and simulation of large complex systems that presently require hundreds of ODEs to simulate and study; (6) their use in understanding the dynamics of complexity; (7) their use in constructing morphologically equivalent systems that can be expressed in closed form in terms of elementary functions. In this respect they provide morphological solutions of ODEs which cannot be solved in closed form in terms of elementary functions, or require conventional numerical methods to solve. For example, there is no closed form solution of the forced Duffing's equation in terms of elementary functions; however, there is an ID solution in terms of elementary functions. (8) Statistical methods and even Stochastic Differential Equations only provide probabilistic correlations between dynamical parameters whereas IDs provide cause and effect relationships between parameters.

The importance of IDs to the study of the morphology of systems is made clear by the human EEG [2]: it is the morphology that determines normal versus clinical status of a human brain. Further, as is demonstrated in evolution, when chaotic systems and events unfold, they only rely on the occurrence of a frequency component rather than the order of occurrence of the frequency component in the dynamic of a phenomena or process. This is the morphology of natural systems. Morphology is nature's way of eliminating the importance of the specificity of the initial conditions in the origination of the dynamics of chaotic or complex process or events. For example, it is well-known that chaotic processes have sensitive dependence on initial condition while still having the same Fourier spectrum. This means that the exact initial conditions are not relevant so long as they are not too far apart, because all chaotic processes which start in a neighborhood of each other lead to the same morphological dynamic. This is the fact that biological and social dynamics depend on for their time evolution: some degree of independence from the initial conditions and the unfolding of the relevant components in any order, which may be random. The example of the tobacco mosaic virus provides a metaphor. If the virus is decomposed into its components and then place in a test tube, it can reassemble itself. Clearly, the order/arrangement in which the components appear in the liquid are not important, but only that they are present and available for a random process to facilitate the reassembly of the virus. IDs provide very direct insight into the morphology of the dynamics of any system. (9) The "laws" on which social and biological systems depend to facilitate the formation of any degree of complexity are stretching and folding. IDs are specifically formulated from these two dynamics and are thus ideally suited to study the morphological dynamics of complex systems. (10) Within the natural world, processes may transition over time from forces that are continuous to forces that become more discrete. IDs can provide the ability to study the dynamics of transition between continuous and discrete, ore even impulsive systems. This study is facilitated by increasing the parameter h of an ID from a small number such as 0.001 to 6.0. This transition is illustrated in Figure 1 for the Simple Scroll [5].

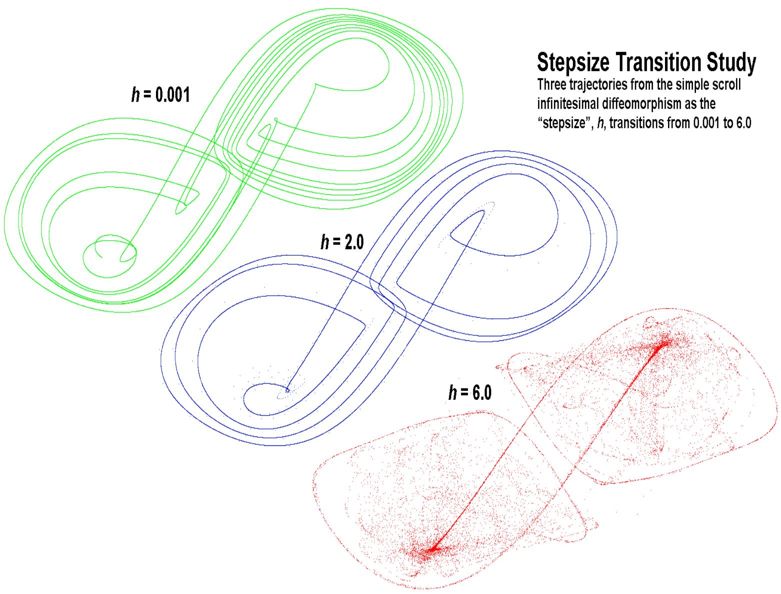

Figure 1: Simple Scroll Transition from Continuous to Discrete

To summarize, IDs are based on the fundamental *complexity* dynamics of biological and social systems rather than the laws of Newton; IDs can be used to predict the dynamics of systems rather than describe the correlation between systems; IDs provide significant computational compression for the formulation of complex biological theories; IDs, for a large class of ODEs of interest to the biological and social sciences, provide very accurate approximations of the solutions of ODEs; IDs are an extension of the concept of dynamical synthesis [8]; IDs are formulated in terms of elementary functions thus allowing for simplicity in modeling, simulation and programming. IDs are formulated as closed-form diffeomorphisms.

2 Stretching and Folding Preliminaries: The Algorithmic Mechanisms, Axioms and Examples

In this section we will cover three preliminaries that will be of use later in the text: (1) How the mechanism of stretching and folding appears algorithmically, (2) Four Axioms for IDs; (3) Two contrasting examples.

2.1 The Algorithmic Form of Stretching and Folding

The Hirsch Conjecture [3] brought attention to the importance of being able to recognize the "form" of stretching and folding as it occurs in a mathematical expression. I will illustrate the algorithmic mechanism of stretching and folding by use of the Henon map. I have modified the Henon map by removing the 1 since is has no bearing on the stretching dynamic [4]. I list the stretching and folding dynamics separately for clarity. The initial condition is the fixed point $(1, 1)$, See Fig 2

$$\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - 2y^2 \\ y \end{pmatrix}, \quad \text{Stretching} \tag{1}$$

$$\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{Folding} \tag{2}$$

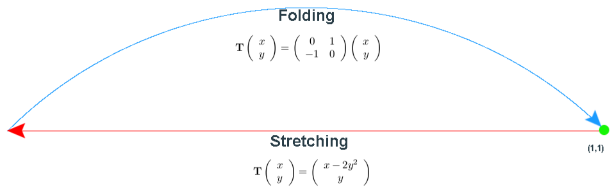

Figure 2: Stretching and folding in the modified Henon map

In Fig.(2) the nonlinear dynamic, Eq.(1), moves a point to the left by an amount related to the magnitude of the y axis coordinate. As y increases, the shift becomes larger creating a shearing effect between to y-levels, see Fig.(3) The folding dynamic Eq. (2) simply rotates a point back in the opposite direction. At a fixed point $(1, 1)$, these two dynamics exactly cancel each other. The complexity of the fixed point is suggested by the lateral direction of the stretching dynamic combined with the somewhat vertical direction of the folding dynamic. If these two dynamics are properly synchronized, it is apparent that the fixed point will be hyperbolic. If not, the fixed point could be elliptic or another fixed point type. What is revealed here is how the action of stretching and folding combine to produce complex dynamics.

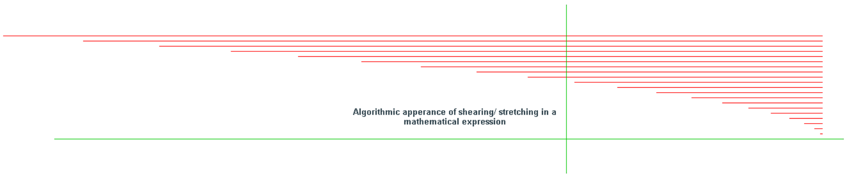

Figure 3: Shearing from Eq.(1) in the Modified Henon Map

2.2 Axioms for IDs

An infinitesimal diffeomorphism $\mathbf{T}_h$ is a parameterized family of smooth mappings on $\mathbf{R}^n$ having the following properties for all $\mathbf{X} \in \mathbf{R}^n$

$$\begin{array}{llcl}
\mathbf{A1} & \|\mathbf{T}_h(\mathbf{X})\| & \leq & M \cdot \|\mathbf{X}\| \;\; \forall\, h \;\; 0 < h \leq 1 \\
\mathbf{A2} & \mathbf{T}_0(\mathbf{X}) & = & \mathbf{X} \\
\mathbf{A3} & \|T_h(\mathbf{X}) - \mathbf{X}\| & \leq & h \cdot M \cdot \|\mathbf{X}\| \\
\mathbf{A4} & \det(\mathbf{J}\mathbf{T}_h(\mathbf{X})) & > & 0
\end{array}$$

Axioms for Infinitesimal Diffeomorphisms

Axiom **A1** says that **T** enjoys a property common to linear transformations; axiom **A3** says that for small h, **T** only moves a point a very short distance, hence the name "Infinitesimal Diffeomorphism", axiom **A4** says that an ID is orientation preserving as are the solutions of ODEs

Typically for the most common applications, h may be thought of as the time variable. For small h, $\mathbf{T}_h^n(\mathbf{X})$ forms an ordered set that closely resembles the orbit of the solution of an ODE since, by axiom **A3**, h may be chosen in order that two consecutive points in the set are arbitrarily close [4]. When an ID is derived from an ODE, the ID provides a very accurate approximation of the solution of the ODE [4]. Also, as seen in [7], IDs can be used to model brain dynamics and to obtain significant computational compression in biological models.

In addition to the above axioms it is useful to have a reference to the "Addition" Axiom

$$\mathbf{A5} \quad \mathbf{T}_h^n \quad = \quad \mathbf{T}_{n \cdot h}$$

This axiom applies to elementary IDs and the solution of ODEs but not to IDs in general.

2.3 A Homogeneous and Inhomogeneous Example

The Homogeneous Case

$$\mathbf{T}_h(\mathbf{X}) = \exp(\mathbf{B} \cdot h)\mathbf{X} \tag{3}$$

where

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{X} \in \mathbf{R}^n. \tag{4}$$

Note that Axiom **A5** applies.

The Inhomogeneous Case

$$\mathbf{T}_h \begin{pmatrix} \mathbf{X} \\ z \end{pmatrix} = \begin{pmatrix} \exp(\mathbf{B} \cdot h)(\mathbf{X} - G(z)) \\ z \end{pmatrix} + \begin{pmatrix} G(z) \\ h \end{pmatrix} \tag{5}$$

where $G : \mathbf{R} \to \mathbf{R}^n$This can also be written as

$$\mathbf{T}_h \begin{pmatrix} \mathbf{X} \\ z \end{pmatrix} = \begin{pmatrix} \exp(\mathbf{B} \cdot h)\mathbf{X} \\ z \end{pmatrix} + \begin{pmatrix} (\mathbf{I} - \exp(\mathbf{B} \cdot h))G(z) \\ h \end{pmatrix} \tag{6}$$

In the inhomogeneous example, the variable z acts like a time variable. Also, Axiom **A5** does not apply.

3 Examples of Infinitesimal Diffeomorphisms of Well-known "Chaotic" ODEs

Example 1: Ueda's equation

$$\ddot{x} + \alpha\dot{x} + x^3 = b\cos(t) \tag{7}$$

where $\alpha = 0.05, b = 7.5$. The objective is to produce a strange attractor using an ID derived from this ODE. First rewrite the equation as a first order two-dimensional system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -\alpha \cdot y - x^3 + b\cos(t) \end{pmatrix} \tag{8}$$

At this point, the time varying forcing function is omitted to get.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -\alpha \cdot y - x^3 \end{pmatrix} \tag{9}$$

Assuming we are only looking at a very small arc over time h RHS is written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -x^2 & -\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{10}$$

Assume for small h a solution of the form

$$\begin{pmatrix} x(h) \\ y(h) \end{pmatrix} = \exp\left(h \cdot \begin{pmatrix} 0 & 1 \\ -x_0^2 & -\alpha \end{pmatrix}\right) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{11}$$

Now rewrite the matrix on the RHS as

$$\exp\left(h \cdot \begin{pmatrix} 0 & 1 \\ -x_0^2 & -\alpha \end{pmatrix}\right) = \exp\left(h \cdot \begin{pmatrix} 0 & 1 \\ -x_0^2 & 0 \end{pmatrix} + h \cdot \begin{pmatrix} 0 & 0 \\ 0 & -\alpha \end{pmatrix}\right) \tag{12}$$

Breaking this down further we get

$$\exp\left(h \cdot \begin{pmatrix} 0 & 1 \\ -x_0^2 & -\alpha \end{pmatrix}\right) = \exp\left(h \cdot \begin{pmatrix} 0 & 1 \\ -x_0^2 & 0 \end{pmatrix}\right) \cdot \exp\left(h \cdot \begin{pmatrix} 0 & 0 \\ 0 & -\alpha \end{pmatrix}\right) \tag{13}$$

which is

$$\begin{aligned} &= \exp(-\alpha \cdot h) \exp\left(h \cdot \begin{pmatrix} 0 & 1 \\ -x_0^2 & 0 \end{pmatrix}\right) && (14) \\ &= \exp(-\alpha \cdot h) \begin{pmatrix} \cos(h \cdot x_0) & \sin(h \cdot x_0)/x_0 \\ -x_0 \cdot \sin(h \cdot x_0) & \cos(h \cdot x_0) \end{pmatrix} && (15) \end{aligned}$$

$$\begin{aligned} x(h) &\approx \exp(-\alpha \cdot h)(x_0 \cdot \cos(h \cdot x_0) + y_0 \cdot \sin(h \cdot x_0)/x_0 && (16) \\ y(h) &\approx \exp(-\alpha \cdot h)(y_0 \cdot \cos(h \cdot x_0) - x_0^2 \cdot \sin(h \cdot x_0)) && (17) \end{aligned}$$

Changing notation and putting the forcing function back into the equation we get

$$\begin{aligned} x_{n+1} &\approx \exp(-\alpha \cdot h)(x_n \cdot \cos(h \cdot x_n) + y_n \cdot \sin(h \cdot x_n)/x_n) \\ y_{n+1} &\approx \exp(-\alpha \cdot h)(y_n \cdot \cos(h \cdot x_n) - x_n^2 \cdot \sin(h \cdot x_n)) + b\cos(n \cdot h) \end{aligned}$$

To obtain a morphologically equivalent attractor we set $\alpha = 0.245$ and $b = 7.5$, see Fig,(4)

Example 2: van der Pol

The ODE

$$\ddot{x} + \alpha \cdot (c - x^2) \cdot \dot{x} + x = a \cdot \sin(\omega \cdot t) \tag{18}$$

Using similar methods as above we derive the ID for the van der Pol equation:

$$\begin{aligned} x_{n+1} &\approx \exp(\alpha \cdot h \cdot (c - x_n^2))(x_n \cdot \cos(h) + y_n \cdot \sin(\omega \cdot h)) \\ y_{n+1} &\approx \exp(\alpha \cdot h \cdot (c - x_n^2))((y_n - b\sin(\beta \cdot n \cdot h)) \cdot \cos(h) - x_n \cdot \sin(\omega \cdot h)) + b\sin(\beta \cdot n \cdot h) \end{aligned}$$

A strange attractor for the van der Pol ID is obtained by using $\alpha = 0.5$, $b = 3.2$, $c = 0.2$, $\omega = 0.3$. $\beta = 0.55$ $h = 2\pi/(4000\beta)$ Fig,(5)

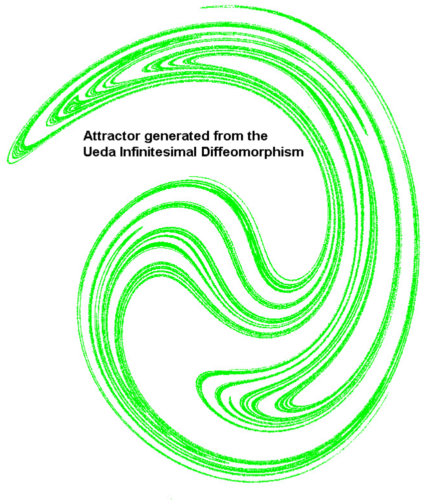

Figure 4: Ueda's Attractor

Example 3: Variation on the Double Pendulum

The ODE

$$\ddot{x} + (\alpha \cdot \dot{x} - x) + x^3 = a \cdot \sin(\omega \cdot t) \tag{19}$$

Using similar methods as above we derive the ID being careful to group the folding and stretching terms together as seen in the equation grouping:

$$\begin{aligned}
ex1 &= (1 - \exp(-\alpha \cdot h))/\alpha \\
ex2 &= c - \exp(-\alpha \cdot h) \\
x_{n+1} &\approx x_n \cdot \cos(h \cdot x_n) + y_n \cdot ex2 \cdot \sin(h \cdot x_n)/x_n + ex1 \cdot \sin(h \cdot x_n) \\
y_{n+1} &\approx ex1 \cdot y_n \cdot \cos(h \cdot x_n) - x_n^2 \cdot \sin(h \cdot x_n) + \\
& \quad ex1 \cdot x_n \cdot \cos(h \cdot x_n)_n + b \sin(n \cdot h)
\end{aligned}$$

A strange attractor for the ID is obtained by using $\alpha = 8.0$, $b = 5.5$, $c = 1.95$, $h = 2\pi/(1000)$ Fig,(6)

Example 4: Morphological Equivalent to the Chua Double Scroll
The dimensionless form of Chua's equations are given by [12]:

$$\begin{aligned}
\dot{x} &= \alpha \,(y - x - f(x)) \\
\dot{y} &= x - y - z \\
\dot{z} &= -\beta y
\end{aligned} \tag{20}$$

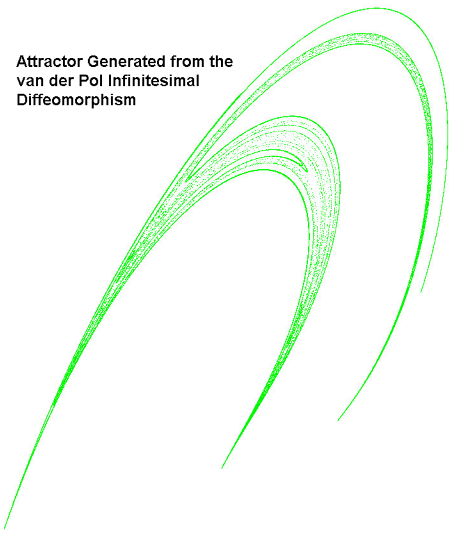

Figure 5: van der Pol Attractor

where,

$$f(x) = \begin{cases} bx + a - b & \text{for} \quad x \geq 1.0 \\ ax & \text{for} \quad |x| \leq 1.0 \\ bx - a + b & \text{for} \quad x \leq 1.0 \end{cases} \tag{2}$$

is a three segment piecewise linear function and α, β are dimensionless parameters. By the methods used above and in [5] we derive the "Chua" ID

$$\begin{aligned} x_{n+1} &= \exp(\alpha \cdot h) \cdot ((x_n - f(u)) \cdot \cos(\omega \cdot h) + y_n \cdot \sin(\omega \cdot h)) + f(u) \\ y_{n+1} &= \exp(\alpha \cdot h) \cdot (y_n \cdot \cos(\omega \cdot h) - (x_n - f(u)) \cdot \sin(\omega \cdot h)) \\ z_{n+1} &= \exp(\beta \cdot f(u) \cdot h) \cdot (z_n - c) + c \end{aligned}$$

where $f(u) = \sin(u) + \sin(3 \cdot u)/3$, $u = x - 3 \cdot z$.

4 The Elementary Stretching Infinitesimal Diffeomorphisms

Stretching and folding are fundamental to the theory of IDs. Folding dynamics are typically provided by the solutions of linear, autonomous ODEs with

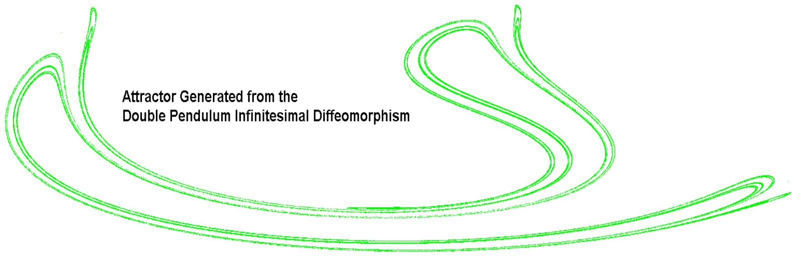

Figure 6: Double Pendulum Attractor

constant coefficients [3] or some form of periodic or almost periodic function. In this section we present four elementary stretching IDs which are related to the solutions of ODEs. To derive the association to ODEs, association we first prove a lemma that tells us when a family of maps solves an ODE [8].

Lemma 1 *Let $g(\mathbf{x}_0, t)$ be a one-parameter family of maps, $\mathbf{x}_0 \to g(\mathbf{x}_0, \cdot)$ from $\mathbf{R}^n$ to $\mathbf{R}^n$ with the following properties:*

(1) For each $\mathbf{x}_0 \in \mathbf{R}^n$, the function $\mathbf{x}(t) = g(\mathbf{x}_0, t)$ is differentiable with respect to t.

(1) $g(\mathbf{x}_0, 0) = \mathbf{x}_0$, for all $\mathbf{x}_0$.

(2)

$$\dot{\mathbf{x}} = \frac{\partial g(\mathbf{x}_0, t)}{\partial t}$$

exists for all $(\mathbf{x}_0, t)$.

(2)$g^{-1}(\mathbf{x}_0, t)$ exists for every t.

Then for fixed $\mathbf{x}_0$, the function $\mathbf{x}(t) = g(\mathbf{x}_0, t)$ solves the initial value problem:

$$\dot{\mathbf{x}} = \frac{\partial g(g^{-1}(\mathbf{x}, t), t)}{\partial t} \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{21}$$

Proof: Since $g^{-1}(\mathbf{x}, t) = \mathbf{x}_0$ for all t we have

$$\dot{\mathbf{x}} = \frac{\partial g(\mathbf{x}_0, t)}{\partial t} = \frac{\partial g(g^{-1}(\mathbf{x}, t), t)}{\partial t}$$

∎

EXAMPLE: Let $n = 1$ and $x(t) = g(x_0, t) = x_0 \exp(t)$. Then for fixed t, $g^{-1}(u, t) = u \exp(-t)$. Hence, $\dot{x} = x_0 \exp(t) = g^{-1}(x, t) \exp(t) = x \exp(-t) \exp(t) = x$.

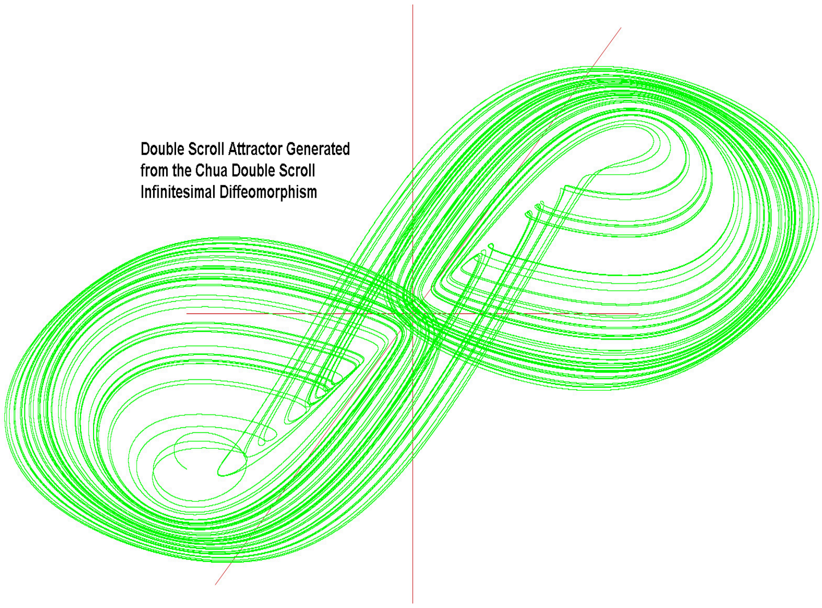

Figure 7: Morphological Equivalent of the Chua Double Scroll

The elementary ID is given by $\mathbf{T}_h(x) = \exp(h) \cdot x$

EXAMPLE: Let

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \exp(t) \\ y_0 - f(x_0) + f(x_0 \exp(t)) \end{pmatrix}$$

Then $\dot{x} = x$ and $\dot{y} = f'(x)\,\dot{x}$. The invertibility requirement means that the initial conditions, (x_0, y_0) can be eliminated from the equation for the derivative and thus we can obtain an equation between the function and its derivative having no arbitrary constants.

The elementary ID is given by

$$\mathbf{T}_h \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \exp(h) \\ y - f(x) + f(x \exp(h)) \end{pmatrix}$$

Lemma 2 *Given a one-parameter family of maps, $g(\mathbf{x}_0, t)$, satisfying the conditions of lemma 1 above and for which*

$$\frac{\partial g}{\partial t}(g^{-1}(\mathbf{x}, t), t)$$

is independent of t, then for $t = h$, the map $\mathbf{T}_h(\mathbf{X}) = g(\mathbf{X}, h)$ is an ID.

Proof: : The ID arises from the solution of an ODE:

$$\dot{x} = \frac{\partial g}{\partial t}(g^{-1}(\mathbf{x}, t), t)$$

. ∎

Let us extend this idea to a finite number of maps satisfying the conditions of lemma 2. In the following lemma, for convenience we use the notation $\mathbf{T}_h =_{def} \mathbf{T}(h)$

Lemma 3 *Given maps $\mathbf{T}_1(h), \mathbf{T}_2(h), \ldots, \mathbf{T}_n(h)$ each satisfying the conditions of lemma 2 the composition of these maps form a ID.*

Proof: The proof consists in applying the definition of ID. ∎

4.1 Example: The Henon Map ID in Three Dimensions

We now apply these ideas to construct an ID which generates the Henon Map Attractor following [8].

Lemma 4 *(*The Henon map*)*
Consider the map,

$$\begin{aligned} x &\rightarrow 1 - ax^2 + y \\ y &\rightarrow b \cdot x \end{aligned}$$

where a, b are constants. There exist an ID which generates the Henon Attractor.

Proof:
The preceding lemmas say that if we are able to factor this map into a composition of maps, each of which is an ID, then their composition is an ID.

We construct this factorization into IDs explicitly along with the one-parameter family of maps, $g(\mathbf{x}_0, t)$, and the ODEs:

1 The factors of the Henon Map:

$$T_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$$

$$T_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 1 - ay^2 \\ y \end{pmatrix}$$

$$T_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ by \end{pmatrix}$$

2 The IDs that are derived from these factors

$$T_1(h)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\pi h/2) & \sin(\pi h/2) \\ -\sin(\pi h/2) & \cos(\pi h/2) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$T_2(h)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + (1 - ay^2)h \\ y \end{pmatrix}$$

$$T_3(h)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \exp(\alpha h) \cdot y \end{pmatrix}$$

3 Three-dimensional Embedding of the Henon Map:

$$\mathbf{T}_1(h)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.5\begin{pmatrix} 1+\cos(\pi h) & 1-\cos(\pi h) & \sqrt{2}\sin(\pi h) \\ 1-\cos(\pi h) & 1+\cos(\pi h) & -\sqrt{2}\sin(\pi h) \\ -\sqrt{2}\sin(\pi h) & \sqrt{2}\sin(\pi h) & 2\cos(\pi h) \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\mathbf{T}_2(h)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + (1 - ay^2)h \\ y \\ z \end{pmatrix}$$

$$\mathbf{T}_3(h)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ \exp(\alpha h)y \\ z \end{pmatrix}$$

For any chaotic two-dimensional map to be an ID it is necessary to embed the map into three-dimensional space. By choosing $a = 1.4$, $b = 0.3$, $t = 1$ and $z = 0.0$ we obtain the familiar Henon attractor in three-dimensional space, [8].

Note that the composition of a finite number of elementary IDs having the same parameter, h, is an ID of parameter h. Also note that an elementary ID, $\mathbf{T}_h$, satisfies Axiom **A5**, hence

$$\mathbf{T}_h \circ \mathbf{T}_h = \mathbf{T}_{2h}$$

Proof is obvious.

4.2 Applying the Method of Dynamical Synthesis Provides the Derivation of Four Elementary Stretching IDs

In this section, we follow the development in [8]. As our focus is to treat IDs that produce complex dynamics, we present four elementary shearing IDs.

The significance of shearing is that it is a form of stretching as was noted in Sec.(2) and it is also a force of nature in both the social and physical sciences. Combining shearing with folding we can obtain all of the commonly recognized chaotic maps such as Henon or Chirikov as was demonstrated in [8]. For example, the Henon map above was decomposed into a shearing force combined with two folding forces.

Lemma 5 (Twist and Shift ID Lemma) *Let*

$$T_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ f(x,y,t) \end{pmatrix}$$

be a one parameter family of C^∞ *invertible maps of* $\mathbf{R}2$ *such that the following is true:*

(1) $T_{s+t} = T_s \circ T_t$, $T_0 = \mathbf{I}$, *the identity map, and* $T_t^{-1} = T_{-t}$*;*

(2) $\det(D(T_t)) = 1$ *for all* t.

Then

$$T_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + \Omega(x)\, t \end{pmatrix}$$

and T_t *is a one-parameter group of twist maps.*

Proof: see [8] From this lemma we get the Twist ID:

$$T_h \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + \Omega(x)\, h \end{pmatrix}$$

The Dilation/Contraction IDs

Proposition 1 (Dilation/Contraction) *Let*

$$T_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ f(x,y,t) \end{pmatrix}$$

be a one parameter family of C^∞ *invertible maps of* $\mathbf{R}2$ *such that the following is true:*

(1) $T_{s+t} = T_s \circ T_t$, $T_0 = \mathbf{I}$, *the identity map, and* $T_t^{-1} = T_{-t}$*;*

(2) $\det(D(T_t)) = \exp(g(x,y,t)) > 0$ *for all* x, y, t.

Then

$$T_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \exp(\Omega(x)\, t) + c(x)(\exp(\Omega(x)\, t) - 1)/\Omega(x) \end{pmatrix}$$

where $c(x)$ *and* $\Omega(x)$ *are* C^∞ *functions of* x *and* T_t *is a one parameter group of dilation/contraction maps. If we are to have this family to be the simplest possible, then* $c(x) = 0$.

Proof: For proof see [8]

The dilation/contraction ID is given by

$$\mathbf{T}_h \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \exp(\Omega(x)\, h) + c(x)(\exp(\Omega(x)\, h) - 1)/\Omega(x) \end{pmatrix}$$

For related discussion see [8].

Let $h = 1/N$ then for an elementary ID $\mathbf{T}$ we have

$$\mathbf{T}_h^{1/h} = \mathbf{T}_h^N = \mathbf{T}_{N\cdot h} = \mathbf{T}_1$$

More complex IDs are derived using the elementary IDs. For example,

Definition 1 Linear Infinitesimal Diffeomorphism
$\mathbf{T}_h$ *is a linear infinitesimal diffeomorphism* (ID) *if there exist a bounded G such that*

$$\mathbf{T}_h(\mathbf{X}) = \exp(\mathbf{A} \cdot h)(\mathbf{X} - G(\mathbf{X})) + G(\mathbf{X})$$

where $G : \mathbf{R}^n \to \mathbf{R}^n$, $\mathbf{X} \in \mathbf{R}^n$, $\mathbf{A}$ is an n by n matrix of constants.

Note that

$$\|\mathbf{T}_h(\mathbf{X}) - \mathbf{X}\| \leq M \cdot h$$

Given an elementary stretching ID $\mathbf{T}_h$, we may construct a more complex ID as follows:

$$\mathbf{X} \to \mathbf{T}_h(\mathbf{X} - G(\mathbf{X})) + G(\mathbf{X})$$

In the above example, the stretching comes from the elementary ID and folding arises from the term $G(\mathbf{X})$. In many cases $G(\mathbf{X})$ is a periodic function. It is possible to reverse this order. For example, in the following case stretching is provided by the term $G(\mathbf{X})$:

$$\exp(\mathbf{A} \cdot h)(\mathbf{X} - G(\mathbf{X})) + G(\mathbf{X})$$

where $\mathbf{A}$ is a constant matrix.

In general, in an ODE, stretching is indicated by the initial conditions occurring nonlinearly in the solution [9, 11]. To make this clear

$$x_0 \cdot \cos(t)$$

is a linear occurrence of an initial condition in part of a solution to an ODE whereas the initial condition appears nonlinearly in the following term in the solution of an ODE:

$$x_0 \cdot \cos(x_0 \cdot t)$$

It is clear that we can construct an ID having a term such as $x \cdot \cos(x \cdot h)$ that is not a part of a solution of an ODE. The occurrence of such a stretching term only needs to make sense for the application.

5 IDs are more General than ODEs

To illustrate the generality of IDs we impose the condition that the Diffeomorphism be measure preserving.

For the conventional linear case $f(t) = \exp(\mathbf{A} \cdot t)\mathbf{X}_0$ the ID is given by the mapping $\mathbf{T}_h(\mathbf{X}) = \exp(\mathbf{A} \cdot h)\mathbf{X}$, where $\mathbf{X}$ is a vector and $\mathbf{A}$ is an n by n matrix. If the solution of the ODE is bounded, then the ID is a very good approximation for even very large h. For $h = 1$ we obtain a finite difference equation.

Consider the following equation:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \cdot \cos(h) + y \cdot \sin(h) \cdot f(x, y) \\ y \cdot \cos(h) - x \cdot \sin(h) \end{pmatrix} \tag{22}$$

For $f = 1$ we have the familiar Linear ID that will approximate the solution to the corresponding ODE. To see how to obtain a much larger range of IDs we require that Det(J($\mathbf{T}$))=1 then we derive the first order PDE

$$\cos^2(h) + y\cos(h)\sin(h)f_x + \sin^2(h)(f + yf_y) = 1 \tag{23}$$

Changing notation to make the equations clear and correspond to conventional solution methods we have

$$\cos^2(h) + y\cos(h)\sin(h) \cdot z_x + \sin^2(h)(z + yz_y) = 1 \tag{24}$$

this gives

$$\begin{aligned} y\cos(h)\sin(h) \cdot z_x + \sin^2(h) \cdot z + \sin^2(h) \cdot y \cdot z_y &= \sin^2(h) && (25) \\ y\cos(h)\sin(h) \cdot z_x + \sin^2(h) \cdot y \cdot z_y &= \sin^2(h) - \sin^2(h) \cdot z && (26) \\ y\cos(h) \cdot z_x + \sin(h) \cdot y \cdot z_y &= \sin(h)(1 - z) && (27) \end{aligned}$$

The solution of this PDE is derived from the relations

$$\frac{dx}{y\cos(h)} = \frac{dy}{y\sin(h)} = \frac{dz}{\sin(h)(1 - z)} \tag{28}$$

from which we obtain $F(x\sin(h) - y\cos(h), (1 - z) \cdot y) = 0$, where F is any arbitrary function of two variables.

Choosing $F(x, y) = x\sin(h) - y\cos(h) + (1 - z) \cdot y = 0$ and solving for z we get

$$z = \frac{x\sin(h) + y \cdot (1 - \cos(h))}{y}$$

Substituting this into Eq.(22) we get

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} (\cos(h) + \sin^2(h)) \cdot x + \sin(h)(1 - \cos(h)) \cdot y \\ y \cdot \cos(h) - x \cdot \sin(h) \end{pmatrix} \tag{29}$$

In practice, F is determined by "boundary conditions" whereas here I have only presented a simple example to illustrate the ideas. To see that this ID does not come from the solution of an ODE note that Axiom **A5** violated:

$$\mathbf{T}_h^2(\mathbf{X}) \neq \mathbf{T}_{2 \cdot h}(\mathbf{X})$$

6 Generalizations of the Example

In this section we present two ideas. (1) IDs can arise from first order PDEs; (2) When using IDs to solve ODEs, it is possible to incorporate first integrals of the ODE to simplify the PDEs.

6.1 An ID, like the solution of a PDE, may contain one or more arbitrary functions

Consider

$$\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \cdot \cos(h) + y \cdot \sin(h) \cdot f(x,y) \\ y \cdot \cos(h) - x \cdot \sin(h) \cdot g(x,y) \end{pmatrix} \tag{30}$$

By requiring that the ID satisfy Axiom **A5**, we can simplify because Axiom **A5** implies that $f \cdot g = 1$. The requirement that Det $(\mathbf{J}(\mathbf{T}))$=1 imposes conditions on f which produce a PDE for f.

$$\begin{aligned}
\mathbf{Det}(\mathbf{J}(\mathbf{T})) &= (\cos(h) + y \cdot \sin(h) \cdot f_x)(\cos(h) - x \cdot \sin(h) \cdot g_y) + \\
&\quad (\sin(h) \cdot g + x \cdot \sin(h) \cdot g_x)(\sin(h) \cdot f + y \cdot \sin(h) \cdot f_y) && (31)\\
&= \cos^2(h) + \sin^2(h) \cdot f \cdot g + \sin^2(h) \cdot x \cdot y(f_x \cdot g_y - f_y \cdot g_x) + \\
&\quad \sin(h)\cos(h)(y \cdot f_x - x \cdot g_y) + \sin^2(h) \cdot (y \cdot g \cdot f_y + x \cdot f \cdot g_x) && (32)\\
&= 1 && (33)
\end{aligned}$$

Note that

$$g_y \cdot f_x - f_y \cdot g_x = 0$$

$$\begin{aligned}
\mathbf{Det}(\mathbf{J}(\mathbf{T})) \quad = \quad & \\
& 1 + \sin(h)\cos(h)(y \cdot f_x - x \cdot g_y) + \\
& + \sin^2(h) \cdot (y \cdot g \cdot f_y + x \cdot f \cdot g_x) = 1
\end{aligned} \tag{34}$$

or, simplifying further

$$\cos(h) \cdot (y \cdot f_x - x \cdot g_y) + \sin(h) \cdot (y \cdot g \cdot f_y + x \cdot f \cdot g_x) = 0 \tag{35}$$

Since $g \cdot f = 1$, one possible solution of this PDE gives the ID

$$\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \cdot \cos(h) + y \cdot \sin(h) \cdot r \\ y \cdot \cos(h) - x \cdot \sin(h)/r \end{pmatrix} \tag{36}$$

Where $r^2 = 0.5(x^2 + \sqrt{x^4 + 4y^2})$ as seen in [9], Sec. 5.2. Equation (36) is a nonlinear autonomous ID that arises from an nonlinear autonomous ODE as seen in [9], sec. 5.2. If we do not impose the addition Axion, **A5**, condition, then a broader range of IDs are obtained. In general, autonomous IDs contain the initial conditions and h with no time variable, and possibly an arbitrary function (to be illustrated later) that is determined by the conditions of the problem to be solved.

6.2 Using a First Integral as a Boundary Condition on the PDEs

Consider

$$\dot{\mathbf{X}} = F(\mathbf{X}) \quad \mathbf{X}(0) = \mathbf{X}_0$$

and assume that there is an invariant function $G(\mathbf{X}) = G(\mathbf{X}_0)$, or first integral, for $\dot{\mathbf{X}}$. Then this relationship can be used in Eq.(30) to place constraints on the system of PDEs that arise from $\mathbf{Det}(\mathbf{J}(\mathbf{T})) = 1$. In particular $G(\mathbf{T}(\mathbf{X})) = G(\mathbf{X})$.

An alternative form of the general solution, $F(x\sin(h) - y\cos(h), (1 - z)\cdot y) = 0$, of the PDE, from Sec. (5) is $(1-z)\cdot y = g(x\sin(h) - y\cos(h))$. Solving for z and changing notation and putting this into Eq.(37) we have

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x\cdot\cos(h) + y\cdot\sin(h)\cdot(y + g(x\sin(h) - y\cos(h))/y \\ y\cdot\cos(h) - x\cdot\sin(h) \end{pmatrix} \tag{37}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x\cdot\cos(h) + y\cdot\sin(h) + \sin(h)\cdot g(y\cos(h) - x\sin(h)) \\ y\cdot\cos(h) - x\cdot\sin(h) \end{pmatrix} \tag{38}$$

Note that sign changes have been made that are irrelevant since g is an arbitrary function. In general, IDs may contain an arbitrary function which, in this case, is eliminated from Eq.(38) by imposing relevant boundary conditions on the problem. If we were to use the first integral $x^2 + y^2 = r^2$ of the simple ID as the boundary condition, then we will find that $g = 0$ and we recover the simple ID Eq.(39)

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x\cdot\cos(h) + y\cdot\sin(h)) \\ y\cdot\cos(h) - x\cdot\sin(h) \end{pmatrix} \tag{39}$$

A routine computation shows that for h=0, that Eq.(38) is the identity map and that it is measure preserving.

7 Linear ODEs with Time Varying Coefficients

Given the equation

$$\dot{\mathbf{X}} = \mathbf{A}'(t)\mathbf{X} \tag{40}$$

Where $\mathbf{A}$ is a square matrix and $\mathbf{X}$ is a vector. Deriving an ID is not as simple as just substituting h for t. The problem is that $\mathbf{A}(t)\mathbf{A}'(t) \neq \mathbf{A}'(t)\mathbf{A}(t)$ in general. A common example is the Bessel's equation of order zero:

$$t\frac{d^2y}{dx^2} + \frac{dy}{dx} + t\cdot y = 0 \tag{41}$$

By substituting a constant for the time variable and solving the linear equation then re substituting the time variable for the constant provides a simple ID that is morphologically equivalent to the Bessel equation, Eq.(41):

$$
\begin{aligned}
x &\rightarrow \exp(\alpha \cdot h/t_n)(x\cos(h) + y \cdot \sin(h)) && (42)\\
y &\rightarrow \exp(\alpha \cdot h/t_n)(y \cdot \cos(h) - x \cdot \sin(h)) && (43)
\end{aligned}
$$

However, it should be expected that the time variable need not occur in the ID. First two short-hand abbreviations:

$$
\begin{aligned}
u &= -h \cdot \log(h) && (44)\\
v &= \sqrt{h/\log(1+h)} && (45)
\end{aligned}
$$

then the time independent ID is given by Equation (46):

$$
\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \cos(u) & \sin(u) \cdot v \\ -\exp(-\alpha \cdot u) \cdot \sin(u) \cdot v & \exp(-\alpha \cdot u) \cdot \cos(u) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (46)
$$

The first approximation ID to Eq. (40) is given by

$$
\mathbf{X} \rightarrow \exp(\mathbf{A}(h) \cdot h)\mathbf{X} \quad (47)
$$

Clearly, this can be generalized and made more precise by the method of Sec. 6

Adding a forcing term can be handled exactly as in the linear nonhomogeneous case treated in [4].

8 Basic Algebraic Relationships for Linear IDs with Constant Coefficients

In addition to the general theory of IDs, there is a need for algebraic relationships that facilitate proofs and simplifications of problems. In this section the subscript designating the parameter h will be omitted to simplify the computations. An ID is linear if the component of Axiom **A3** solves a linear ODE.

Definition

$$
\mathbf{T}^G(\mathbf{X}) = \exp(\mathbf{A} \cdot h)(\mathbf{X} - G(\mathbf{X})) + G(\mathbf{X}) \quad (48)
$$

then $\mathbf{T}^0(\mathbf{X}) = \exp(\mathbf{A} \cdot h)\mathbf{X}$, where $0(\mathbf{X}) = 0$ for all $\mathbf{X}$ and $S_G(\mathbf{X}) = \mathbf{X} - G(\mathbf{X})$.

Theorem 1 *Sums*
Let G, F be mappings from $\mathbf{R}^n$ to $\mathbf{R}^n$ and $(G+F)(\mathbf{X}) = G(\mathbf{X}) + F(\mathbf{X}))$ then

$$
\mathbf{T}^{G+F} = \mathbf{T}^G + \mathbf{T}^F - \mathbf{T}^0 \quad (49)
$$

Proof:

$$\begin{aligned}
\mathbf{T}^{G+F} &= \exp(\mathbf{A}\cdot h)(\mathbf{X}-G-F)) + (G+F) && (50)\\
&= \exp(\mathbf{A}\cdot h)(\mathbf{X}-G) + G + F - \exp(\mathbf{A}\cdot h)(F) && (51)\\
&= \mathbf{T}^{G} + F - \exp(\mathbf{A}\cdot h)(F) && (52)\\
&= \mathbf{T}^{G} + F - \exp(\mathbf{A}\cdot h)(F) + \mathbf{T}^0 - \mathbf{T}^0 && (53)\\
&= \mathbf{T}^{G} + \mathbf{T}^{F} - \mathbf{T}^0 && (54)
\end{aligned}$$

This theorem is easily generalized:

$$\mathbf{T}^{\sum F_i} = \sum_i \mathbf{T}^{F_i} - (n-1)\cdot\mathbf{T}^0$$

This result allows the decomposition of a very complex sum into smaller components.

Theorem 2 *Compositions*
Let G, F be mappings from $\mathbf{R}^n$ to $\mathbf{R}^n$ and $(G\circ F)(\mathbf{X}) = G(F(\mathbf{X}))$ then

$$\mathbf{T}^{G\circ F} = \mathbf{T}^{G}(F) + \mathbf{T}^{F} - F \tag{55}$$

Proof:

$$\begin{aligned}
\mathbf{T}^{G\circ F} &= \mathbf{T}^0(S^{G\circ F}) + G\circ F && (56)\\
&= \mathbf{T}^0(S^{F} + S^{G}(F)) + G\circ F && (57)\\
&= \mathbf{T}^0(S^{F}) + \mathbf{T}^0(S^{G}(F)) + G\circ F && (58)\\
&= \mathbf{T}^0(S_{F}) + \mathbf{T}^0(S^{G}(F)) + F - F + G\circ F && (59)\\
&= \mathbf{T}^0(S_{F}) + F + \mathbf{T}^0(S^{G}(F)) + G\circ F - F && (60)\\
&= \mathbf{T}^{G}(F) + \mathbf{T}^{F} - F && (61)
\end{aligned}$$

Theorem 3 *Nonlinearity*
Let G be a mapping from $\mathbf{R}^n$ to $\mathbf{R}^n$ with $\mathbf{X}, \mathbf{Y} \in \mathbf{R}^n$ and let $\Lambda(\mathbf{X},\mathbf{Y}) = G(\mathbf{X}+\mathbf{Y}) - (G(\mathbf{X}) + G(\mathbf{Y}))$ (note that Λ measures the degree to which G is nonlinear) then

$$\mathbf{T}^{G}(\mathbf{X}+\mathbf{Y}) - (\mathbf{T}^{G}(\mathbf{X}) + \mathbf{T}^{G}(\mathbf{Y})) = (\mathbf{T}^0 - \mathbf{I})\Lambda(\mathbf{X},\mathbf{Y}) \tag{62}$$

Proof: Direct computation.

Theorem 4 *Conjugation*
Assume $\mathbf{T}^{G}(\mathbf{A})(\mathbf{X}) = \exp(\mathbf{A}\cdot h)(\mathbf{X} - G(\mathbf{X})) + G(\mathbf{X})$ then

$$\mathbf{T}^{G}(\mathbf{J}^{-1}\mathbf{A}\mathbf{J})(\mathbf{X}) = \mathbf{J}^{-1}\mathbf{T}^{\mathbf{J}G}(\mathbf{A})(\mathbf{J}\mathbf{X}) \tag{63}$$

Proof: Direct computation.

Theorem 5 *Diagonalization*
Assume $\mathbf{T}^G(\mathbf{A})(\mathbf{X}) = \exp(\mathbf{A}\cdot h)(\mathbf{X} - G(\mathbf{X})) + G(\mathbf{X})$ *where* $\mathbf{A} = \mathbf{D} + \mathbf{N}$ *and* $\mathbf{D}\cdot\mathbf{N} = \mathbf{N}\cdot\mathbf{D}$ *then*

$$\mathbf{T}^G(\mathbf{D}+\mathbf{N})(\mathbf{X}) = \mathbf{T}_0(\mathbf{N})(\mathbf{T}^G(\mathbf{D})(\mathbf{X}) = \mathbf{T}^0(\mathbf{D})(\mathbf{T}^G(\mathbf{N})(\mathbf{X}) \tag{64}$$

Proof: Direct computation. This result and the former apply to expressing $\mathbf{A}$ in Jordan Normal Form.

9 An Inventory of IDs

The following is a listing of the most common IDs. While the form of an ID can be algorithmically complex, there are two forms that are common. Let $\mathbf{T}_t(X_0))$ be the solution of an ODE with initial condition $\mathbf{T}_0(X_0) = X_0$, then

$$\mathbf{X} \to \mathbf{T}_h(\mathbf{X} - G(\mathbf{X})) + G(\mathbf{X})$$

is an ID for smooth G. Next,

$$\mathbf{X} \to \mathbf{T}_h(\mathbf{X}) + H(h, \mathbf{X})$$

is an ID for smooth H.

Linear Homogeneous ID

$$\mathbf{T}_h(\mathbf{X}) = \exp(\mathbf{A}\cdot h)\mathbf{X} \tag{65}$$

Origin

$$\dot{\mathbf{X}} = \mathbf{A}\cdot\mathbf{X} \tag{66}$$

Axiom **A1** of Sec. (2), requires that the absolute value of the eigenvalues of $\mathbf{A}$ is less than or equal to 1. Clearly, $\mathbf{T}_0 = \mathbf{I}$.

$$\|\mathbf{T}_h(\mathbf{X}) - \mathbf{X}\| = \|\exp(\mathbf{A}\cdot h)\mathbf{X} - \mathbf{X}\| \leq \|\mathbf{A}\|\cdot h \tag{67}$$

Linear Inhomogeneous

$$\mathbf{T}_h(\mathbf{X}) = \exp(\mathbf{A}\cdot h)(\mathbf{X} - G(\mathbf{X})) + G(\mathbf{X}) \tag{68}$$

Origin

$$\dot{\mathbf{X}} = \mathbf{A}\cdot(\mathbf{X} - H(\mathbf{X})) + H(\mathbf{X}) \tag{69}$$

Note that $H \neq G$ but is derived to assure that the ID is morphologically equivalent to the solution of the ODE.

Simple Nonlinear Homogeneous

$$\mathbf{T}(\mathbf{X}) = \exp(f(\mathbf{X})\mathbf{A} \cdot h)\mathbf{X} \tag{70}$$

where f is a complex valued function of $\mathbf{X}$. If f is a constant along integral curves, this ID originates from

$$\dot{\mathbf{X}} = f(\mathbf{X})\mathbf{A} \cdot \mathbf{X} \tag{71}$$

Origin The twist equation of [9], Sec. 5.1 is an example.

Simple Nonlinear Inhomogeneous

$$\mathbf{T}(\mathbf{X}) = \exp(f(\mathbf{X})\mathbf{A} \cdot h)(\mathbf{X} - G(\mathbf{X})) + G(\mathbf{X}) \tag{72}$$

If f is a constant along integral curves, this ID originates from

$$\dot{\mathbf{X}} = f(\mathbf{X})\mathbf{A} \cdot (\mathbf{X} - H(\mathbf{X})) + H(\mathbf{X}) \tag{73}$$

Origin The square wave Twist and Flip equation from [10], Sec. 3.2 is an example.

Compound Nonlinear Homogeneous

$$\mathbf{T}(\mathbf{X}) = \exp(\mathbf{A}(\mathbf{X}) \cdot h)\mathbf{X} \tag{74}$$

Special case

$$\mathbf{T}(\mathbf{X}) = \exp(\mathbf{A}(f(\mathbf{X})) \cdot h)\mathbf{X} \tag{75}$$

Where f is constant along integral curves.

Origin This ID originates from [9], Sec. 5.2.

$$\dot{\mathbf{X}} = \mathbf{A}(f(\mathbf{X})) \cdot \mathbf{X} \tag{76}$$

Compound Nonlinear Inhomogeneous

$$\mathbf{T}(\mathbf{X}) = \exp(\mathbf{A}(\mathbf{X}) \cdot h)(\mathbf{X} - G(\mathbf{X})) + G(\mathbf{X}) \tag{77}$$

Origin An example is the morphological equivalent of the Chua double scroll, Fig. (7).

10 Fundamental Conjectures

Conjecture 1 *Assume that*

$$\dot{\mathbf{X}} + \frac{\partial \mathbf{A}(\mathbf{X}, t)}{\partial t} \cdot \mathbf{X} = b(t) \tag{78}$$

has a unique bounded solution for each initial condition, then there exist an ID *that morphologically approximates the solution of* Eq.(78) *to any arbitrary degree of accuracy.*

Guess:

$$\mathbf{X}_{n+1} = \exp(\mathbf{A}(\mathbf{X}_n, h) \cdot h)(\mathbf{X}_n - b(t_n)) + b(t_n) \tag{79}$$

Conjecture 2 *Consider the finite difference equation:*

$$\Delta \mathbf{T}(\mathbf{X}) = F(\mathbf{T}(\mathbf{X})) \tag{80}$$

There exist an ID *which, for* $h = 1$*, is equal to* $\mathbf{T}$

Conjecture 3 *Let* $\mathbf{T}(\mathbf{S}, \mathbf{F})$ *be an ID with stretching and folding components* $\mathbf{S}, \mathbf{F}$ *The there exists an algebraic relationship*

$$G(\mathbf{S}, \mathbf{F}) = 0$$

which determines when a transverse homoclinic point exists exist for $\mathbf{T}$.

11 Summary

In this paper I have introduced the formal concept of the Infinitesimal diffeomorphism. I have discussed why a formal study of Infinitesimal Diffeomorphisms as a separate mathematical discipline is important for social and biological sciences as well as for engineering. Why IDs differ from the solutions of ODEs is presented and why IDs do not rely on the laws of physics for their formulation and use has been discussed. I note that IDs are based on the dynamics of complexity, which is stretching and folding and that IDs provide an avenue to construct morphological equivalent solutions of equations when the morphology is all that is relevant such as in the case of the human EEG. Also, IDs provide cause and effect relationships between systems rather than just correlations as is down by using statistical methods. Thus IDs provide the potential to efficiently model human systems on a level that is not possible using ODEs or even PDEs. Unlike FDEs in which a very large step size is used, IDs provide a variable step size parameter that allows for simulations of discrete systems to any given level of detail. IDs can also provide dramatic compression of the algorithms of models that are based on ODEs because IDs capture dynamics in a manner similar to Gaussian integration. It is shown that IDs are more general than ODEs and thus more general than systems based on the laws of physics. I have also given a glimpse of how IDs relate to well known strange attractors in the study of chaos. A particularly valuable point for modeling and simulation is that IDs are close-form equations that are expressed in terms of elementary functions. When an ID is derived from an ODE, the ID can provide a very accurate approximation of the solution of the ODE even for very large step size. IDs also provide a more direct window into the complex dynamics of ODEs than is presently available.

12 References

[1] Smale S., [1967] "Differentiable Dynamical Systems", *Bull Am Math Soc* **73** 747-817.

[2] Henon,M., [1976] "A Two-dimensional Mapping with a Strange Attractor", *Commun. math. Phys.* 50, 6977 (1976)

[2] Brown, R., Jain, V. [2009] "A New Approach to Chaos", *Dynamics of Continuous, Discrete and Impulsive Systems* Series A Mathematical Analysis 16 863-890

[3] Brown, R. [2014] "The Hirsch Conjecture", to appear in *Dynamics of Continuous, Discrete and Impulsive Systems*

[4] Brown, R. [2014] "Infinitesimal Stretching and Folding Part I" to appear

[5] Brown, R. [2014] "ISF Part II: The Simple Scroll" to appear

[7] Brown, R and Hirsch, M.W., "Stretching and Folding in the KIII Neurodynamical Model", to appear

[8] Brown, R., Chua, L. [1993] "Dynamical Synthesis of Poincaré Maps", *International Journal of Bifurcation and Chaos* **3**(5),1235-1267]

[9] Brown, R., Chua, L. [1998] "Clarifying Chaos II: Bernoulli Chaos, Zero Lyapunov Exponents, and Strange Attractors", *International Journal of Bifurcation and Chaos* **8**(1), pp.1-32

[10] Brown, R. Chua, L. [1991] "Generalizing the Twist-and-Flip Paradigm", *International Journal of Bifurcation and Chaos* **1**(1), 385-416

[11] Davis, H.T., [1962] *Introduction to Nonlinear Differential and Integral Equations*, Dover Publications, New York

[12] Brown, R. [1992] "Generalizations of the Chua Equations", *International Journal of Bifurcation and Chaos* **2**(4), 889-909

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