Reverse Engineering Turbulence

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Abstract

This paper derives a model of turbulence by *reverse engineering* or, equivalently, dynamical synthesis. Also, a derivation of a climate parameter is presented. In both cases, the resulting equations contain stretching and folding which underlies the dynamics of chaos and, therefore, greatly limits the long-term predictability of fluid flow models that are derived from the Naiver-Stokes equations.

Keywords. Chaos, turbulence, dynamical systems, dynamical synthesis

AMS (MOS) subject classification: 37D45.

Contents

1 Introduction

Turbulence cannot be rigorously defined. However, the Office of Naval Research identifies three properties of turbulence (to be referred to as ONR1): (1) disorder; (2) efficient mixing; (3) vorticity. Ergodic theory provides some simplification by combining the highest level of disorder and efficient mixing into a single dynamical system, the bilateral shift. Thus ONR1 may be restated, in the most extreme case, as the level of turbulence that combines a bilateral shift with vorticity. The significance of the bilateral shift to turbulence is that the bilateral shift is the mathematical idealization of a coin toss experiment and is therefore the essence of unpredictability.

If this is a valid description, then constructing a vortex having a bilateral shift should have some recognizable characteristics common to turbulent fluid flow. A convenient example of turbulent flow is supplied by the wake of a large jet aircraft. Demonstrating that a vortex having a bilateral shift can produce a similar visual image through its time one map to the wake of a jet, while not a mathematical proof, is persuasive. See Fig. 1.

Figure 1: Plate A is the turbulent wake of a jet; Plate B is the time one map of a periodically driven chaotic system

The simplest formulation of a bilateral shift on two symbols is captured in the following equation:

$$\left.\begin{array}{rcl} u &=& 2x-[2x] \\ v &=& (y+[2x])/2 \\ x &=& u \\ y &=& v \end{array}\right\} \tag{1}$$

where $[z]$ is the integer part of real number z.

The first line of Eq. 1 is a unilateral shift. All of the complexity of the shift is found in this line. It has two parts: (1) and exponential product, $2x$, and a periodic part, $[2x]$ also referred to as a $\mathrm{mod}(1)$ operation. This abstract formulation can be replaced by the baker's transformation [2] on the unit square.

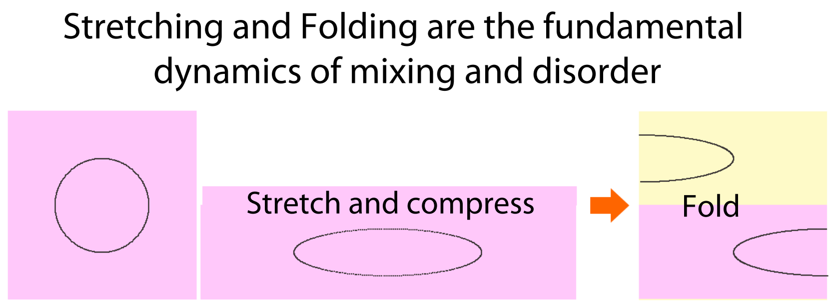

Figure 2: This figure illustrates the dynamics of Eq. 2

$$g(x,y) = \begin{cases} (2x, y/2) & \text{mod}(1),\ \text{if}\ 0.0 \leq x < 0.5 \\ (2x, (y+1)/2) & \text{mod}(1),\ \text{if}\ 0.5 \leq x < 1.0 \end{cases} \quad (2)$$

Figure 2 also illustrates the dynamics of the Smale horseshoe [3], page 771.

In order to understand how the shift may appear in nature, a dynamical system must be derived whose time one map is the shift. This is done in [4]. An illustration of this diffeomorphism is seen in Fig. 3 derived from the baker's transformation.

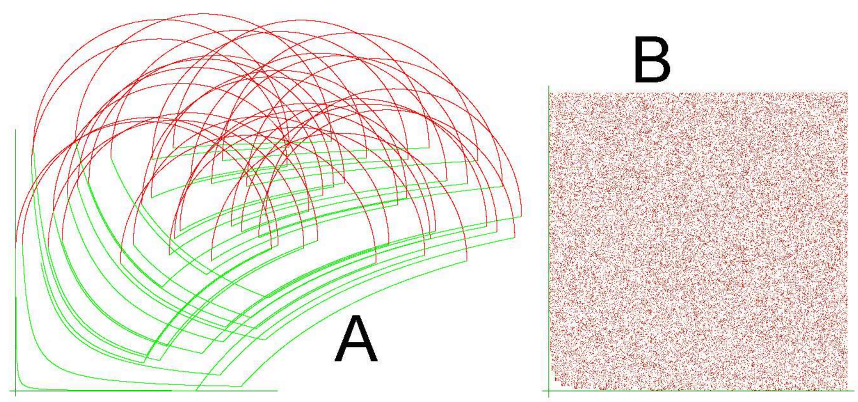

Figure 3: Plate A is the trajectory of a single particle from the dynamics of the bilateral shift; Plate B is The Time-one map for Plate A

In Fig. 3 Plate A, is illustrated the path of a single particle of a bilateral shift on two symbols. Plate B is the time one map of this trajectory. ONR1 specifications 1 and 2 are illustrated in this figure. What is not present is vorticity. Vorticity may be added by adding curl. This will be addressed in Sec. 2.

The significance of Fig. 3 is that the particle trajectories are as random as Brownian motion. In fact, the bilateral shift is the basis for formulating a mathematical model of a white noise stochastic process [4].

2 Reverse Engineering a Particle Trajectory

This section will add vorticity to the baker's transformation, Eq. 2. The starting point is a simplified Euler equation.

$$\left.\begin{array}{rcl} \dot{x} & = & f(r)y \\ \dot{y} & = & -f(r)x \end{array}\right\} \tag{3}$$

where $f(r)$ is some function of the radius $r = \sqrt{x^2+y^2}$. Equation 3 can be solved in closed form in terms of elementary functions. The solution can be expresses in IDE form with the use of the matrix

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4}$$

The solution of Eq. 3 is, in IDE form,

$$\mathbf{T}_h = \exp(hf(r)\mathbf{B}) \tag{5}$$

Figure 4, Plate A has ten orbits of Eq. 3. In Plate B, the analog of a mod(1) dynamic is added. If a particle is moving along a trajectory of the Euler equation and then is acted upon by a separate dynamic that forces the particle back into the the interior of the vortex, Plate B is obtained. The blue circle is a trajectory of a particle moving along the streamline of the vortex and the orange segment is the result of the particle being forced back into the vortex by a separate dynamic. The next step is to formulate this mathematically. The guiding principle is the concept of the horseshoe, or stretching and folding, as seen in Fig. 2

The stretching dynamic can be an exponential or can be the interface between a nonlinear rotation and a linear rotation as seen in [**?**]; the $\mathrm{mod}(1)$ part can be found in a periodic dynamic.

In IDE form, the periodic dynamic is

$$\mathbf{S}_h = \exp(h\mathbf{B}) \tag{6}$$

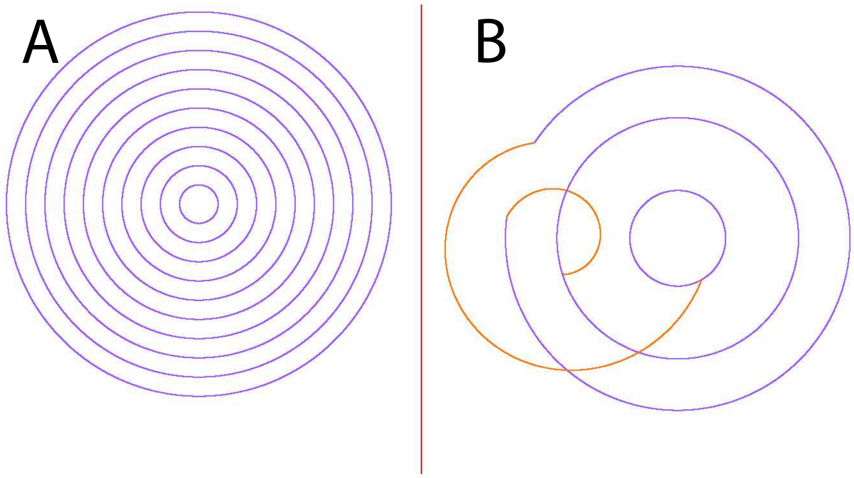

Figure 4: Plate A is the Time series for 10 orbits for Eq. 3 ; Plate B is The orbit for one point that combines Plate A and a mod(1) action using a periodic function

As written, the orbits of $\mathbf{T}_h$ and $\mathbf{S}_h$ coincide. To create turbulence (using the ONR1 specification) they must be offset. This is done by shifting Eq. 5 off center by A. Doing this gives

$$\mathbf{T}_h(\mathbf{X}) = \exp(h\,f(r)\,\mathbf{B})\,(\mathbf{X}-A)+A \tag{7}$$

with $r = \sqrt{(x-1)^2+y^2}$

Where

$$A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

To combine these two separate IDE dynamics into a single IDE a boundary condition must be introduced that determines how these two dynamics interact. This combination is Eq. 8 where $s(\mathbf{X})$ is the boundary condition. See [4] for the code and the exact boundary condition.

$$s(\mathbf{X})\cdot(\exp(h\,f(r)\,\mathbf{B})\,(\mathbf{X}-A)+A)+(1-s(\mathbf{X}))\cdot\exp(h\,\pi\mathbf{B}) \tag{8}$$

After combining these two separate simple IDEs into a single IDE according to the theory found in [4], Fig. 5, Plate A is obtained for the trajectory of a single particle. Plate B is a series of 120,000 snap shots, at a macro-scale level of this trajectory taken at equally spaced time intervals, the time-one map, projected onto a two-dimensional plane. At the macro scale, the fluid nature of this time-one plot becomes visible.

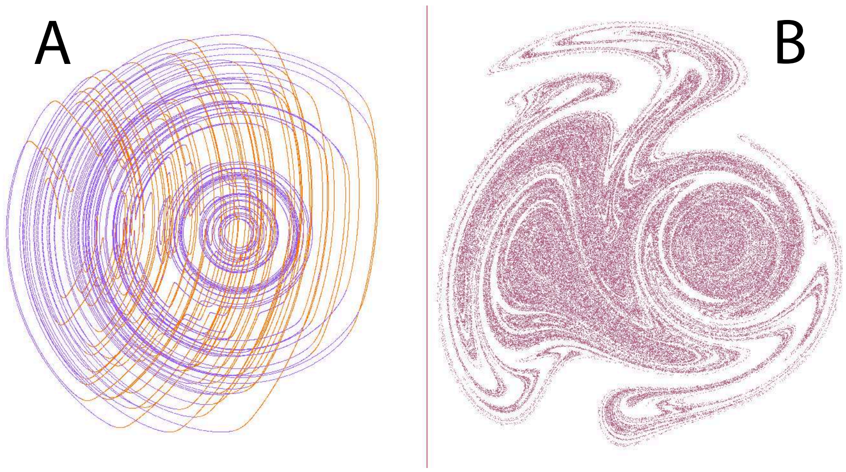

Figure 5: Plate A is the trajectory of a single particle driven by Eq.8; Plate B is 120,000 points of the macro scale time-one plot for Plate A

Various fluid-like macro scale time one maps may be constructed by varying $f(r)$, see Fig. 6.

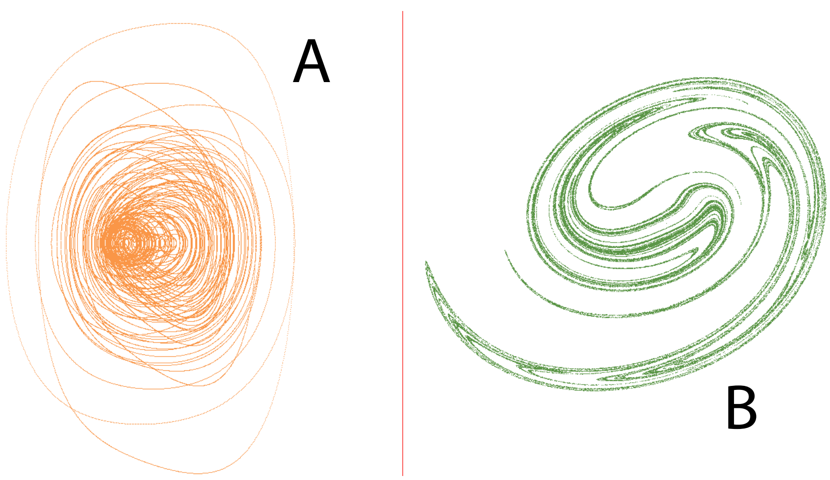

Figure 6: Plate A is the trajectory of a single particle driven by Eq.8 with $f(r) = (\sin(1/r)^2) \cdot 2.75\gamma/r$; Plate B is 120,000 points of the macro scale time-one plot for Plate A

The code for these images may be found in [4].

The conclusion that can be reached by the reversed engineered trajectory equations of small-scale turbulence is that inherent in turbulence is a function of a bilateral shift and as a result, predicting the particle path is like predicting a series of coin tosses.

3 Reverse Engineering a Climate Parameter

Following Saltzman [5] and Lorenz [6] an IDE is derived as in [4] that is a local solution to the system of their ODEs. Equation 9 is the result of that derivation. This system if the composition of infinitesimal folding followed by infinitesimal stretching.

Figure 7, Plate A is the graph of a solution to the Saltzman-Lorenz equations. The solution may represent any atmospheric parameter.

Figure 7: Graphs of the Saltzman-Lorenz Equations derived from Eq. 9. Plate A is the time series; Plate B is the time one map of Plate A

$$\left.\begin{aligned} h &= 0.001 \\ u_1 &= x \\ v_1 &= y \cdot \cos(x \cdot h) + z \cdot \sin(x \cdot h)/1.5 \\ w_1 &= z \cdot \cos(x \cdot h) - y \cdot \sin(x \cdot h) \cdot 1.5 \\ \\ x &= \exp(-3 \cdot h) \cdot (u_1 \cdot \cosh(2\pi \cdot h) + v_1 \cdot \sinh(2\pi \cdot h)) \\ y &= \exp(-0.04 \cdot h) \cdot (v_1 \cdot \cosh(2\pi \cdot h) + u_1 \cdot \sinh(2\pi \cdot h)) \\ z &= \exp(-1.6 \cdot h) \cdot w_1 \end{aligned}\right\} \quad (9)$$

The Saltzman-Lorenz equations provide insight into predicting the climate at some point in the future based on present measurements. Even if there existed exact solutions of a climate model such as GCM and that the present state could be measured with only very small errors, the best that can be done is to make predictions for a very short time, days at most, because the errors would magnify rapidly (exponentially). For example, in the case of tornadoes (for which there is an enormous amount of initial data about the parent super cell), the predictions are wrong 70% of the time and this is for time-scales of minutes. As another example, if a tropical storm forms off the west coast of Africa, can its exact course be predicted on the basis of the tropical storm's state at the time of formation? This is impossible today and may be theoretically impossible as proven by Poincaré in the late 1800s. Thus, predicting climate fluctuation/change years in the future is mathematically impossible. The problem lies in the nature of processes that have locally exponential dynamics, as climate models do due to their dependence on the Naiver-Stokes equations. They are no more predictable than a coin toss as seen in Fig. 7, Plate B.

4 Summary

In this paper, by reverse engineering, equations for the time evolution of a particle in a turbulent flow are derived and the equations for the time evolution of a climate parameter are derived. Underlying both derivations is the role of stretching and folding, and, therefore, the role of functions of the bilateral shift [4] in limiting predictability of the dynamics of fluid flow as the unilateral and the bilateral shift are the mathematical realization of a coin toss experiment.

It is also known that stretching and folding combine to produce chaos through the mechanism of a transverse homoclinic point [3] thus providing a link between fluid flow and chaos.

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